PROCEDURES FOR SOLVING DIFFERENTIAL EQUATIONS
(explicit form of solution is y = , other forms are called implicit forms)

<table>
<tr><th colspan="4">FIRST ORDER DIFFERENTIAL EQUATIONS</th></tr>
<tr><th>NAME/SECTION</th><th>PATTERN</th><th>PROCEDURE</th><th>EXAMPLES, NOTES</th></tr>
<tr><td>INTEGRATING FACTOR METHOD (sec 2.1)</td><td>$\frac{dy}{dt} + p(t)y = g(t)$

(remember that other variables besides y and t can be used)</td><td>1) Put equation into pattern.
2) Find the integrating factor as $\mu(t) = e^{\int p(t)dt}$
3) Multiply all terms by $\mu(t)$ and check that the left side equals $(\mu y)'$ using product rule
4) Integrate both sides, don't forget the +c on the right.
5) Solve for y = .</td><td></td></tr>
<tr><td>SEPARABLE EQUATION (sec 1.2, 2.2)</td><td>M(x) dx = N(y) dy</td><td>1) Put equation into pattern.
2) Integrate both sides, putting +c on one side only.
3) Solve for y = , if possible.</td><td></td></tr>
<tr><td>HOMOGENEOUS EQUATION (sec 2.2 ex. #30)</td><td>$\frac{dy}{dx}$ = only constants and $\frac{y}{x}$ forms</td><td>1) Put equation into pattern.
2) Let $v = \frac{y}{x}$, $y = xv$, and $\frac{dy}{dx} = v + x\frac{dv}{dx}$
3) Solve like other separable equations.
4) Sub in $\frac{y}{x}$ for v and simplify.
5) Solve for y = if possible.</td><td></td></tr>
<tr><td>EXACT EQUATION (sec 2.6)</td><td>$M(x,y) + N(x,y)\frac{dy}{dx} = 0$</td><td>1) Put equation into pattern.
2) Check if $M_y = N_x$. If ≠, look for possible integrating factor (see next procedure).

3) Find $\psi = \int M dx$ using h(y) for constant.
4) Find $\psi_y = N$ to solve for h(y)

3) Find $\psi = \int N dy$ using h(x) for constant.
4) Find $\psi_x = M$ to solve for h(x)

5) $\psi(x,y) = c$ is the solution</td><td></td></tr>
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NAME/SECTION	PATTERN	PROCEDURE	EXAMPLE, NOTES
INTEGRATING FACTOR FOR EXACT EQUATION (sec 2.6, example 4, exercise #23, exercise #24)	$M(x,y)+N(x,y)\frac{dy}{dx}=0$ but $M_y \neq N_x$	**1) Find** $\frac{N_x - M_y}{M} = Q$. **2) If Q is a function of y only, the integrating factor is:** $\mu(t) = e^{\int Q(y)dy}$. **OR** **1) Find** $\frac{M_y - N_x}{N} = Q$. **2) If Q is a function of x only, the integrating factor is:** $\mu(t) = e^{\int Q(x)dx}$. **OR** **1) Find** $\frac{N_x - M_y}{(xM - yN)} = Q$. **2) If Q is a function of xy only, the integrating factor is:** $\mu(t) = e^{\int Q(x,y)d(xy)}$. **3) Multiply by the integrating factor and complete using Exact Equation Method.**	

SECOND AND HIGHER ORDER DIFFERENTIAL EQUATIONS			
NAME/SECTION	**PATTERN**	**PROCEDURE**	**EXAMPLE, NOTES**
HOMOGENEOUS EQNS WITH CONSTANT COEF (sec 3.1 & 4.2)	$ay'' + by' + cy = 0$ **(extendable to higher order equations)**	**Convert to** $ar^2 + br + c = 0$**, called the characteristic equation** **For 2 real roots:** $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ **For 2 complex roots** $\lambda \pm \mu i$**:** $y_1 = e^{(\lambda + \mu i)t}$ $y_2 = e^{(\lambda - \mu i)t}$ **(sec 3.1)** $y_1 = c_1 e^{\lambda t} \cos \mu t$ $y_2 = c_2 e^{\lambda t} \sin \mu t$ **(sec 4.1)** **For repeated roots of any type:** $y_2 = t y_1$ $y_3 = t^2 y_1$ **etc.**	**n^{th} of a complex number: (r is negative)** $\sqrt[n]{r} = \sqrt[n]{\lvert r \rvert}\left[\cos\left(\frac{\pi}{n} + \frac{2\pi}{n}k\right) + i \sin\left(\frac{\pi}{n} + \frac{2\pi}{n}k\right)\right]$ **for** $k = 0, 1, 2, \ldots, n-1$
LINEAR INDEPENDENCE & DEPENDENCE (sec 3.2 & 3.3)	**Wronksian of solutions (extendable to higher order equations)**	**y_1 and y_2 are potential solutions for ODE** $w = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ **(determinant)** **If** $w \neq 0$**, y_1 and y_2 are independent; else they are dependent.** **For 3rd order ODE, can find Wronksian as:** $w = ce^{-\int p(t)\,dt}$**, from** $y'' + p(t)y' + q(t)y = 0$	
REDUCTION OF ORDER (sec 3.5)	**ODE with non-constant coefficients and one given solution**	**1. let $y = vy_1$** **2. Find y' and y'' for y.** **3. Substitute in to original ODE. Cancel and collect like terms. Will get form in v'' and v'** **4. Let z = v' and z' = v''. Will get 1st order eqn.** **5. Solve by separation or other method and put back into v form.** **6. Again, let $y = vy_1$**	

NAME/SECTION	PATTERN	PROCEDURE	EXAMPLE, NOTES
NONHOMOGENEOUS EQN WITH UNDETERMINED COEF (sec 3.6, 3.7)	$y'' + p(t)y' + q(t)y = g(t)$	see below	see below

PROCEDURE

General Method:

1. Find y_c as solution to homogeneous eqn.

2. Find particular solution Y of nonhomogeneous equation using one of three methods below.

3. Put together as $y = y_c + Y(t)$

Undetermined Coefficients (Superposition)

1. Look at g(t) and guess form (see right).
If y_c already contains that form, multiply form by t, t^2, etc. as needed.

2. Find Y, Y', Y'' and substitute into original equation.

3. Determine coefficients by matching them to g(t) coefficients.

Variation of Parameters

$$u_1 = \int \frac{y_2\, g(t)}{W(y_1, y_2)}\,dt \qquad u_2 = \int \frac{y_1\, g(t)}{W(y_1, y_2)}\,dt$$

$$Y(t) = -y_1 u_1 + y_2 u_2$$

Wronksian Method (not in text)

$$W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 & y_2 \\ g(t) & y_2' \end{bmatrix} \quad W_2 = \begin{bmatrix} y_1 & 0 \\ y_1' & g(t) \end{bmatrix}$$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W} \qquad \text{integrate to find } u_1 \text{ and } u_2$$

$$Y = u_1 y_1 + u_2 y_2$$

EXAMPLE, NOTES

Pattern	Possible form for Y(t)
any constant	A
3t+8	At + B
$4t^2$-5	At^2 + Bt + c
t^3+2t-3	At^3 + Bt^2 + Ct + D
sin 3t	A cos 3t + B sin 3t
cos 3t	A cos 3t + B sin 3t
e^{2t}	A e^{2t}
(3t +4)e^{2t}	(At + B) e^{2t}
t^2 e^{2t}	(At^2 + Bt + C) e^{2t}
e^{2t}sin 3t	Ae^{2t} cos 3t + Be^{2t} sin 3t
$4t^2$sin 3t	(At^2 +Bt + C)cos 3t + (Et^2 + Ft + G)sin 3t
te^{2t} cos 3t	(At + B)e^{2t}cos 3t + (Ct + D)e^{2t}sin 3t