PARTIAL FRACTION DECOMPOSITION

USE: rational functions when degree of numerator < degree of denominator.
If degree of numerator $\geq$ degree of denominator, do polynomial division first.

A. DISTINCT LINEAR FACTORS: example: $\frac{5x-2}{x^3-4x}$

1. Factor and separate denominator. Numerators are variables representing constants, denominators are linear expressions (factors of original denominator).

ex. $$\frac{5x-2}{x^3-4x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

2. Multiply both sides by least common denominator (LCD).

ex. $$5x-2 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

3. Function has domain of $\Re - \{0, -2, 2\}$ so substitute those values in for x.

ex. for $x = 0$ equation in step 2 becomes $-2 = -4A$ so $A = \frac{1}{2}$
for $x = -2$ equation in step 2 becomes $-12 = 8B$ so $B = -\frac{3}{2}$
for $x = 2$ equation in step 2 becomes $8 = 8C$ so $C = 1$

4. Substitute A, B, and C into equation in step 1 to get:

ex. $$\frac{5x-2}{x^3-4x} = \frac{\frac{1}{2}}{x} + \frac{-\frac{3}{2}}{x+2} + \frac{1}{x-2} = \frac{1}{2x} - \frac{3}{2x+4} + \frac{1}{x-2}$$

B. REPEATED LINEAR FACTORS: example: $\frac{2x}{(x-1)^3}$

1. Decompose denominator into progression of linear factor(s) with increasing exponents up to original exponent. Numerators are variables representing constants.

ex. $$\frac{2x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

2. Multiply both sides by LCD. ex. $2x = A(x-1)^2 + B(x-1) + C$

3. Original equation has domain of $\Re - \{1\}$ so substitute 1 in for x.

ex. for $x = 1$ equation in step 2 becomes $2 = C$

4. Substitute any other numbers you wish into equation of step 2 to get equations involving A and B. Use easy numbers like ±1, 0, etc. Result will be two equations with two variables. Solve these equations simultaneously to get values for A and B.

ex. for $x = -1$ and $C = 2$, equation in step 2 becomes $-2 = 4A - 2B + 2$

ex. for $x = 0$ and $C = 2$, equation in step 2 becomes $0 = A - B + 2$

Solving these two equations simultaneously gives $A = 0$ and $B = 2$

5. Substitute A, B, and C into equation in step 1 to get:

ex. $$\frac{2x}{(x-1)^3} = \frac{0}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3} = \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3}$$

C. DISTINCT LINEAR AND QUADRATIC FACTORS: example: $\frac{x^2+3x-1}{(x+1)(x^2-2)}$

1. Decompose denominators by factoring. Linear denominators have constant numerators, quadratic denominators have linear numerators.

ex. $$\frac{x^2+3x-1}{(x+1)(x^2-2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2}$$

2. Multiply both sides by LCD. ex. $x^2+3x-1 = A(x^2-2) + (Bx+C)(x+1)$

3. Substitute integer values not in domain

ex. for $x = -1$ equation in step 2 becomes $3 = A$

4. Substitute any other numbers you wish into equation of step 2 to get values for B and C. Use easy numbers like ±1, 0, etc.

ex. for $x = 0$ and $A = 3$, equation in step 2 becomes $C = 5$

ex. for $x = 1$, $A = 3$, and $C = 5$, equation in step 2 becomes $B = -2$

5. Substitute A, B, and C into equation in step 1 to get:

ex. $$\frac{x^2+3x-1}{(x+1)(x^2-2)} = \frac{3}{x+1} + \frac{-2x+5}{x^2-2}$$

D. ALTERNATIVE METHOD FOR LINEAR AND QUADRATIC FACTORS

Do #1 and #2 as above (Linear and Quadratic Factors)

3. Multiply out equation from step #2 and collect like terms:

$$x^2+3x-1 = Ax^2 - 2A + Bx^2 + Bx + Cx + C$$

equate coefficients of like powers on left and right

$$1x^2+3x-1 = (A+B)x^2 + (B+C)x + (C-2A)$$

4. so $i: 1 = A+B,\ ii: 3 = B+C,\ iii: -1 = C-2A$

Solve as simultaneous equations to find $A = 3,\ B = -2,\ C = 5$

5. Substitute A, B, and C in equation from step 1 above.